Understanding Numbers

The National Council of Teachers of Mathematics states that in grades 3-5 all students should:

- understand the place-value structure of the base-ten number system and be able to represent and compare whole numbers and decimals
- recognize equivalent representations for the same number and generate them by decomposing and composing numbers

Principles and Standards for School Mathematics p. 148
National Council of Teachers of Mathematics

Students who are proficient in mathematics and have the foundational understandings necessary for future success have a deep understanding of place value. They know more than how to tell which digit is in the hundreds place; they know more than how to use an algorithm to get an answer to a computation problem. They understand the structure of the numbers and what happens to the numbers when they add, subtract, multiply, or divide. They understand, for example, that 2,345 is composed of 2 groups of a thousand, 3 groups of one hundred, 4 groups of ten, and 5 ones. They can easily reorganize ones into tens (e.g. 42 ones is 4 tens and 2 ones), tens into hundreds (e.g. 17 tens is 1 hundred and 7 tens), and hundreds into thousands (e.g. 15 hundreds is 1 thousand and 5 hundreds). They use what they know about whole numbers to help them understand decimals. They see that 42 hundredths is worth 4 tenths and 2 hundredths. They know that 7 tens and 5 tens can be combined to form 1 hundred with 2 tens left over and use that information to combine 7 tenths and 5 tenths. They know what quantities are represented by numbers so can easily tell that .30 is smaller than .5.

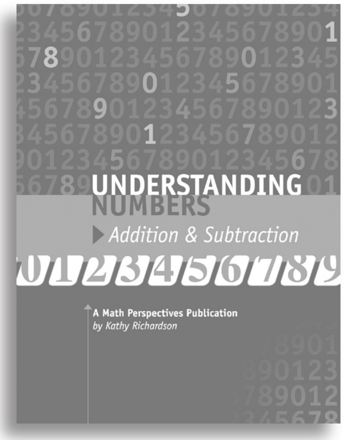
Students who understand the underlying structure of the numbers work with numbers with facility and ease and have the knowledge needed to develop competence with computation.

Developing An Understanding of the Structure of Numbers

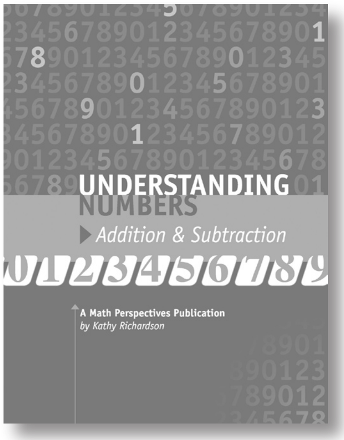
Developing procedures for the addition and subtraction of multi-digit numbers, both whole numbers and decimal numbers, should evolve from the students' understanding of place value.

Principles and Standards for School Mathematics p. 218
National Council of Teachers of Mathematics

Students need many ongoing experiences to develop an understanding of the underlying structure of numbers and the ability to think about numbers flexibly. They need to engage in activities where they group and regroup numbers, break numbers apart, and reorganize them in various ways. They need to find out for themselves that 12 tens, 120 ones, and 1 hundred and 20 ones all describe the same quantities. They need to actually build .12 and 1.2 to perceive the relationship between them.



Understanding Numbers



Understanding Numbers

The Stations

The Understanding Numbers series of stations are sets of 8 tasks that provide students with the meaningful practice necessary for developing an understanding of the underlying structure of numbers, number relationships and operations. The mathematics presented in the stations is foundational and crucial for developing computational fluency. These foundational concepts also provide the basis for students' understanding of the mathematics they will encounter in the future. Each set of stations presents a variety of activities focused on one major concept. The tasks are designed to meet a range of needs allowing all students to work at their own level. These stations should be experienced over and over again until students have developed proficiency with the tasks. Most students will benefit from working with the appropriate set of stations for several weeks.

Using Models

Students develop an understanding of the structure of numbers by working with models that reveal relationships between numbers. They develop proficiency with the numbers represented by the models when their focus is on identifying the relationships the models reveal rather than on manipulating the models to get answers. They focus on these relationships when they organize and reorganize various models into hundreds, tens, and ones or into tenths and hundredths. So, for example, instead of trading 10 tens for 1 hundred (which can cloud the relationship for some students) they reorganize the tens into 1 hundred. Instead of trading 1 hundred for 10 tens so they can take away 4 tens, they mentally take away or cover up the 4 tens to see that 6 tens are left. They don't count by tens to see what 14 tens are worth, but rather they reorganize the tens into 1 hundred and 4 tens to determine that the blocks are worth 140. When combining 8 and 6, they don't trade 10 ones for a ten, but rather they mentally or physically put 2 of the ones with the 8 to make another 10 and see that 4 are left.

Work with models is effective if it leads students to a level of understanding where they can mentally work with the ideas represented by the models and therefore no longer need the models.

"Research indicates that students' experiences using physical models to represent hundreds, tens, and ones can be effective if the materials help them think about how to combine quantities and, eventually, how these processes connect with written procedures ...The models, however, are not automatically meaningful for students; the meaning must be constructed as they work with the materials."

Adding it Up, p. 198
National Research Council

Differentiating the Tasks to Meet a Range of Needs

... a very important part of the job of a teacher is to guide the child towards tasks where he will be able objectively to do well, but not too easily, not without putting forth some effort, not without difficulties to be mastered, errors to be overcome, creative solutions to be found. This means assessing his skills with sensitivity and accuracy, understanding the levels of his confidence and energy, and responding to his errors in helpful ways."

Margaret Donaldson
Children's Minds p. 120

Students develop a full understanding of important concepts at various times. If each student is to learn all they can, each one must be working at the edge of their own understanding as they work to develop proficiency. The various tasks included in the sets of stations, therefore, are expandable. That is, they can be adapted to meet a range of instructional needs.

Sometimes, having students work with tens and ones rather than with hundreds, tens, and ones, while other students work with numbers in the thousands meets this range of needs. Sometimes the ways that students use models automatically differentiates the task. For example, some students may need to move the blocks or touch the models to solve a problem. Others may need to study the models to aid their thinking but will not need to actually move or touch the model. Others will be able to do the task without the use of a model at all.

Interacting with the Students

Students do not discover or understand mathematical concepts simply by manipulating concrete materials. Mathematics teachers need to intervene frequently as part of the instruction process to help students focus on the underlying mathematical ideas and to help build bridges from the students' work with the manipulatives to their corresponding work with the mathematical ideas or actions.

Jerry Johnson
Teaching and Learning Mathematics, p. 40

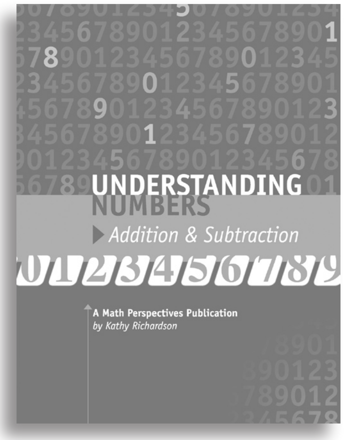
Just as books are the tools that students use to learn to read, the sets of stations are tools for learning mathematics. It is not the task itself that determines what students learn, but rather, what the students are focused on while doing the task that influences the learning. Students will be working with the station tasks independently. However, the teacher has an important role in helping students focus on the important mathematics so that the students learn all they can from their work with the various tasks. The teacher needs to interact with the students while they are at work, observing how they solve problems, questioning them as a way of focusing their thinking, and challenging them. Each station includes suggestions for interacting with students at work.

Introducing the Stations

The whole class can work on these stations at the same time. The stations should be introduced to the whole class over a period of 3 or 4 days. The directions for most of the task are simple and easy for the students to understand. After the tasks have been introduced and the students have had some time to work with the stations, the teacher can begin to adapt the task to more appropriately meet the various needs. Suggestions for adapting the tasks are included in the Notes for Teachers for each task.

Readiness to Work with Stations

Students will be more successful working with the tasks if a culture of self-directed learning and hard work has been established. The students should know how to choose a task, work hard at the task, and move to another task without direction from the teacher. It is worth devoting some time to the development of this way of working.



Understanding Numbers

Addition/Subtraction Stations

Goals:

The students will learn to:

- Combine quantities without counting, by reorganizing tens into hundreds, and ones into tens to find the totals
- Break apart and recombine hundreds, tens, and ones when adding and subtracting
- Demonstrate the relationships between numbers using Base Ten Blocks and hundreds grids
- Add by splitting numbers into parts that make them easier to add
- Add by changing numbers to make them easier to add and then adjusting the sums accordingly
- Subtract by changing numbers to make them easier to subtract and then adjusting the differences accordingly
- Use a number line to determine the differences between numbers
- Explain their strategies using words, drawings, and symbols

The goal of the Multi-Digit Addition and Subtraction Stations is to support students' development of computational fluency. The National Council of Teachers of Mathematics defines computational fluency in the following manner:

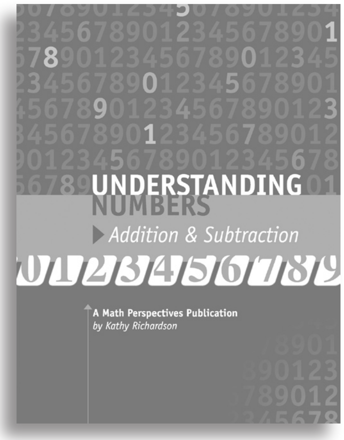
Computational fluency refers to having efficient and accurate methods for computing. Students exhibit computational fluency when they demonstrate flexibility in the computational methods they choose, understand and can explain these methods, and produce accurate answers efficiently. The computational methods that a student uses should be based on mathematical ideas that the student understands well, including the structure of the base-ten number system, properties of addition and subtraction and multiplication and division, and number relationships.

Principles and Standards for School Mathematics p. 152
National Council of Teachers of Mathematics

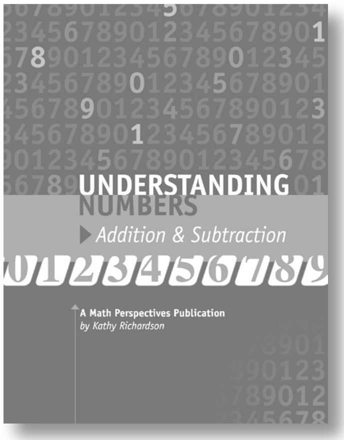
When students compute using mathematical ideas they understand well, they know more than how to "borrow and carry" following the step-by-step procedures of standard algorithms. Instead of solving every problem in the same way, they recognize the particular relationships inherent in the numbers they are working with and arrive at their answers in efficient and logical ways. For example, when adding 198 + 87, they might move 2 from the 87 to make 200 and immediately see that they have 285. When subtracting 26 from 100, they might use what they know about money and almost instantly see that they need to take 1 quarter and 1 more away to arrive at 74. When solving the problem 1002 – 426, they might set the 2 aside and take 400 from 1000, leaving 600. They could then use what they know about subtracting from 100 and easily subtract 26 from 600 arriving at 574. They would add the 2 back on to arrive at the answer of 576.

The procedures children construct on their own build directly on the foundational number concepts, and these underlying concepts often are quite visible when one examines the step in the procedures.

Adding It Up, p. 152
National Research Council



Addition/Subtraction Stations
Introduction



Addition/Subtraction Stations

Introduction

When students are limited to a memorized procedure for getting answers, the underlying mathematics inherent in the algorithm is often not visible to them. The National Research Council's Committee on Mathematics Learning explains why learning to "borrow and carry" is so difficult. "Standard algorithms, in contrast to children's constructed algorithms, are quite far removed from their conceptual underpinings. They have evolved over centuries for efficiency and compactness. They can be executed quickly but they can be difficult to learn with understanding." (Adding it Up, p. 201). However, students who understand the underlying structure of numbers, notice the relationships in the numbers, and use this knowledge to compute are actually using more mathematics than students who solve problems using memorized procedures.

Learning to use algorithms for computation with multi-digit numbers is an important part of developing mathematical proficiency. Algorithms are procedures that can be executed in the same way to solve a variety of problems arising from different situations and involving different numbers. Children can and do devise algorithms for carrying out multi-digit arithmetic, using reasoning to justify their inventions and developing confidence in the process.

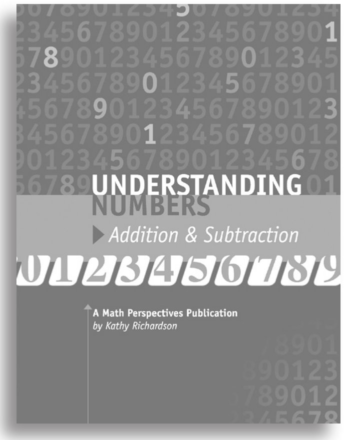
Adding It Up, p 7
National Research Council

The tasks in this set of stations give students a variety of experiences that help them learn the mathematics necessary for facility with multi-digit addition and subtraction. The tasks are designed to make certain relationships more apparent and to give students practice using particular strategies.

Developing Efficient Strategies

When students begin their work with addition and subtraction, most do not use efficient strategies. They use whatever mathematics they know thus far. The tasks in this set of stations present students with certain types of problems that lend themselves to particular strategies. This alone does not guarantee that they will understand and be able to use those strategies. It is the teacher's job to help students focus on and understand the underlying mathematics in each of the station tasks. It is through repeated experiences, questioning by the teacher, and discussions with other students that students will come to understand the mathematics and find "easier and easier" ways to solve problems.

When students are considering various ways to solve problems, it is important that they solve problems in ways that make sense to them. Consider the task, *Roll A Number*, for example. In this task, students add and subtract numbers ending in 9. There is the expectation that students will learn that changing the 9s to 10s can make the problems easier to solve. However, this should not be required since this strategy may not yet make sense to some students. They may not think about 9 in terms of its relationship to 10. They may find it confusing when deciding whether to add 1 or take 1 away from their answer. The teacher might nudge the student to consider changing 9s to 10 by asking questions such as, "What if the problem was 27 + 10 — would that be harder or easier than adding 9?" "If we added 10 to this, would we be adding too much or not enough?" "What do we have to do to the answer if we want to add 9 instead of 10?" These questions may help some students see that



Addition/Subtraction Stations
Introduction

changing 9s to 10 is a "good idea" for solving the problem and they will want to use that strategy on their own. Some students will need to see the problems acted out with models in order to understand what is happening to the numbers and may or may not adopt this as a reasonable way to solve problems. Other students, however, may not have reached the level of thinking that allows them to use this strategy with understanding. For them, changing the number changes the problem and they cannot see how this can be helpful. The teacher should try to help the students focus on particular relationships and strategies but must not require students to solve problems in particular ways. It is more important for students to use whatever makes sense to them than it is to learn a new strategy they do not fully understand.

Several of the tasks in this set of stations focus on determining the relationships or differences between numbers. Learning to find the differences between numbers is particularly difficult for many students but is critical to developing proficiency with subtraction. Practice with models in a variety of settings will help students understand what is meant by "difference" and how to determine the difference in more than one way.

Sometimes students will find the difference between numbers such as 73 and 46 by figuring out what needs to be taken away to end up with 46.

For example:
73- **20** = 53   53 - **3** = 50   50 - **4** = 46

When they add up all that they have taken away, they find they need to take 27 away from 73 to end up with 46.

20 + 3 + 4 = **27**   73 - **27** = 46

Sometimes students will count up from one number to another to determine the difference. In this case, they find out what they need to add to 46 to get to 73.

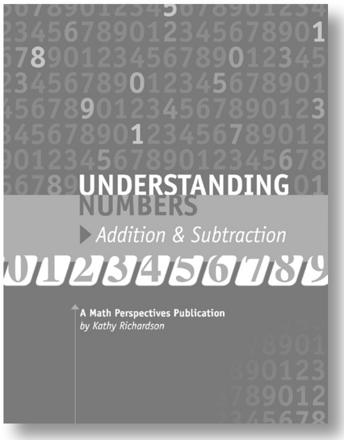
For example:
From 46 to 66 is **20**. From 66 to 70 is **4**. From 70 to 73 is **3**.

When combining what has been added on, they find that they needed to add 27 to 46 to get to 73.

20 + 4 + 3 = **27**   46 + **27** = 73

Comparative language is difficult for some students to interpret. Students do not always know what is being asked when they hear "How many more?" and "What is the difference?" In order to help students understand what is being asked, the teacher can use slightly different language. Instead of asking "How many more?", the teacher could ask, "How many extras would there be?" or "How many would be left over?" For example: "There are 326 children in the school and 345 cartons of milk. How many extra cartons of milk will there be if every child gets one carton of milk?" Later, after these types of situations are understood, the connection to the particular phrase "How many more?" can be made.

Students can be helped to understand what is being referred to by the difference if they are asked to determine what is the same about the numbers before they find the difference.



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A problem such as the following can be presented to help highlight this idea. "Peter has 15 coins in his collection. Joli has 21. Joli has the same number as Peter plus some more. How many more does Joli have than Peter."

If necessary, counters representing the coins can be matched up to see what's the same and what's different.

Peter: ●●●●●●●●●●●●●●●
Joli: ●●●●●●●●●●●●●●●●●●●●●

Finding what is the same about the numbers and taking that away in order to see what is different can help students interpret the language and also help them make the connection between the mental actions they use when solving these kinds of problems and the subtraction equation they are asked to use. In this case, Joli would take away 15 of her coins to see how many more she has than Peter. She is using subtraction (21-15) in a meaningful way to find the difference.

Using Models

Models are used in the Addition and Subtraction Stations to help students see the structure of numbers, how numbers are related, and what happens when numbers are broken apart and/or combined. The models are intended to help students check their thinking and, importantly, should lead to the ability to solve problems without models. When students use a variety of models, they learn that what happens to the numbers when they add or subtract can be represented in a variety of ways. In this set of stations, students work with Base Ten Blocks, various grids representing thousands, hundreds, tens, and ones, as well as number lines. When students use the Base Ten Blocks, they should combine numbers by reorganizing ones into tens, tens into hundreds and hundreds into thousands. They should break apart numbers mentally or by covering them up rather than trading. This keeps the focus on the relationships rather than on procedures for getting answers.

Prerequisites

Even though the Addition and Subtraction Stations are designed so students can work with them at many different levels and all students should be able to work with them with some success, those students who have a strong understanding of place value will have the necessary foundation to learn the most from these tasks.

Research has shown that it is difficult to develop procedural fluency with multi-digit arithmetic without an understanding of the base-10 number system. If such understanding is missing, student make many different errors in multi-digit computations."

Adding it Up, p. 199
National Research Council

Teachers can help students extend their understanding of addition and subtraction with whole number to decimals by building on a solid understanding of place value.

Principles and Standards for School Mathematics p. 218
National Council of Teachers of Mathematics

If you have students who do not have a strong understanding of place value, they would benefit from experiences with the Place Value Stations from the Understanding Numbers series of stations.

Ongoing Experiences with the Stations

It is through ongoing, repeated experiences with the stations that students develop an understanding of and facility with the various strategies used in multi-digit addition and subtraction. They move through various stages when working with these tasks. In the beginning some may count to get answers. With practice, they will learn to use what they know about the structure of the numbers to add and subtract. Eventually, they will be able to do the task mentally or symbolically without the use of models. The students will benefit from ongoing experiences with these tasks until they can do the tasks without the use of models.

Interacting with the Students

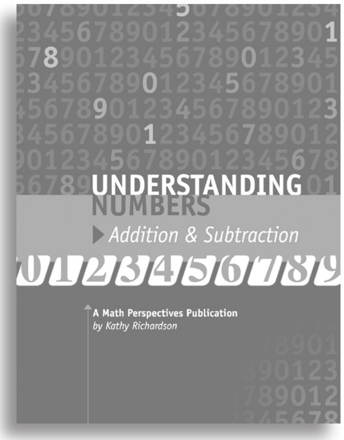
Students work independently with the various station tasks, choosing the particular tasks they want to do. The teacher's role during this time is vital. The teacher observes the students while they are at work and interacts with them, asking questions, focusing and challenging their thinking. For example, when the teacher sees that a student is counting to add instead of combining groups, she asks questions instead of just telling the student not to count. The questions engage the student in thinking about the numbers. The teacher might ask, "If you put all those tens together, would you have enough to make a hundred and some leftover tens?" Some students will show evidence that they still need to count. Other students will discover they do know how to get answers without counting but were relying on counting "just to make sure." The teacher's questions are to help the students notice relationships but should not be used to get the students to do a task in one particular way. If any students are unable to do what the teacher expects, it is important to allow them to do it in the way that makes sense to them.

When doing the tasks in this set of stations, the students are often asked to keep track of the steps they take when solving a problem. If this is difficult for a student, the teacher can have the student describe orally what he or she is doing while the teacher does the recording of the steps. After the teacher has modeled this for a time, the student will be able to do the recording on his or her own.

The Notes to the Teacher for each of the stations includes suggestions for interacting with students. There is also a *Guide to Observations Card* that lists suggested questions. This can be kept handy as a reminder for teachers during their conversations with the students.

Adapting the Task

Each of the tasks can be done at three levels: using 2-digit numbers, 3-digit numbers and 4-digit numbers. If teachers find that students are not efficient and count much of the time to get answers, they should have them work with smaller numbers. If students can add and subtract 3-digit numbers with ease without the use of the model, they should move to adding and subtracting 4-digit numbers.



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