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# FOREWORD

This book was written to help Pre-K though 4th educators recognize the complexities of the mathematics young children are expected to learn, and to identify what is required for children to develop an understanding of number concepts. The information is also useful for those who work with older children who do not yet have the foundation necessary for success.

The Critical Leaning Phases, identified here, are not skills that can be directly taught to children. Rather, they are the understandings that must be in place if children are going to be successful in the study of mathematics. Children develop understandings of the Critical Learning Phases through instruction that builds on what they already know and understand.

The Critical Learning Phases are often unrecognized or assumed to be in place as children move through the math curriculum. This means many children do not have key prerequisite understandings in place before they receive instruction. They may appear to be successful in the short term but their lack of meaningful learning eventually shows up. This leads to many children believing they are failures in math. Much of their time and energy is spent on trying to learn certain procedures when what they really need is the foundation on which the procedures are based.

Children who do not expect math to make sense simply guess or try to remember what the teacher said. These children look at the teacher to see if they are right, rather than to their own ability to think and make sense.

Children will learn MORE in the long run when they are provided the instruction they need, rather than given extra time to study what they are not yet able to learn with meaning. Children who expect math to make sense will puzzle over what they don't understand, ask questions and try things out to see what happens. They are persistent and work to make sense. I have yet to meet a child who is not "tickled" when they figure something out that challenges them—but is also within reach.

Teachers can maximize what children learn if they know what level of thinking they have developed, and what they still need to understand regarding a particular concept. Teachers will be able to recognize the difference between getting their students to do or say something that gives the appearance of knowledge and evidence that shows they really know.

It is possible to meet each child's needs within a classroom setting even though the children are not all working at the same level. This is because within whatever larger concept the children are learning, there are levels of understanding. So for example, if a class is working on comparing numbers, some children will be able to find the differences between numbers to 20, and others will need to work with numbers to 10 and identify only which number is more and which is less than the other. If children are working with place value, some children will be just beginning to understand how to count tens as units, and other children will be combining numbers to make all the tens they can.

Just as children read at different levels, children can work on math concepts at different levels. What is key is that the child is making real progress and not trying to do what they are told without insight.

# ACKNOWLEDGEMENTS

This work is the culmination of more than 40 years working with children and teachers: observing, wondering, discussing, reading, and thinking. I have been helped, challenged and enlightened by interactions with so many over the years that it is impossible to list all who have been part of this journey.

This is to all the children and teachers who accompanied me along the way and to my family who gave me undying support every step of the way.

Thanks to all of you.

Kathy

# Introduction

## The Critical Learning Phases:

*Key to Effective Instruction*

# THE CRITICAL LEARNING PHASES:
## Key To Effective Instruction

I have been intrigued with the study of how children learn mathematics for more than forty years, beginning when I taught my first class of 36 unforgettable first graders.

Since then, I have worked with children from pre-school through sixth grade, and in the field of special education—both in my own classroom, and alongside teachers who shared their students with me. Over the years, I have interacted with children in many regions of this country and others. I have analyzed children's responses, seeking to delve deeper into their thinking to determine how they perceived the problems I gave them, and how they interpreted various mathematical situations.

I found there are crucial mathematical ideas that students must understand if they are to find meaning in the mathematics they are expected to learn. I call these understandings "Critical Learning Phases." The term "critical" is used to underscore the essential role each level or phase has on how children make sense of numbers and thus, what mathematics they are able (or not) to learn. Critical Learning Phases are the understandings that must be in place to ensure that children are not just imitating procedures or saying words they don't really understand. They are milestones, or hurdles, in children's growth of understanding; they are insights, rather than facts or procedures.

Children can learn to answer questions and follow procedures, yet lack the level of awareness that gives real meaning to the math. For example, a child may be able to line up 12 counters, carefully touch each one, and tell the teacher how many she counted. But we cannot say she understands what she needs to know about counting unless she can also count out 12 objects. Can she keep the 12 in mind or will she count right past it? Another child may have memorized "6 and 6 is 12," but can he use what he knows to solve 6 + 7? A child may have learned that 3 tens and 4 is 34, but if 10 more are added, can he tell how many there are without counting?

Whether a child understands a Critical Learning Phase can be determined by asking questions or observing behaviors. A child who understands a Critical Learning Phase finds the answer(s) to be obvious. Children who do not yet understand respond in ways that reveal their level of thinking.

When children are taught mathematical concepts or procedures before they reach certain levels of thinking, they do not see the underlying logic of the mathematics they are working with. All they can do is memorize processes and procedures. It may appear that they know the mathematics, but in reality, this is just an illusion. What they have learned is not useful to them because they cannot build on it. This “illusion of learning” breaks down at the point where true understanding is necessary for further growth.

If we are to ensure that children have learned the foundational mathematics they need as they move on to higher levels, educators must be aware of the development of the Critical Learning Phases and the indicators that reveal what children do and do not understand. If we look only at the ability to get right answers, we miss the information needed

to determine what children know and still need to learn. The result is that children spend valuable instructional time trying to memorize what doesn't make sense to them, instead of developing the understandings they need.

The Critical Learning Phases presented here focus on number concepts. The development of number concepts is the foundation and heart of the mathematics program for young children. What children know and understand about number and number relationships impacts every other area of mathematical study. Children cannot analyze data, determine functional relationships, compare measures of area and volume, or describe relative lengths of sides unless they can use numbers in meaningful ways. Number concepts are the foundation that children must have in order to achieve high standards in mathematics as a whole.

Children who understand number concepts know that numbers are used to describe quantities and relationships, and are useful tools for getting information about the world they live in. They see relationships between numbers, and can take numbers apart and put them back together without counting. They understand the structure of numbers and think of numbers as made up of groups of tens and ones, or hundreds, tens and ones, and so forth. They know what happens to the numbers when they add, subtract, multiply, or divide. They use symbols to represent numerical ideas, and can explain and interpret what the symbols mean. They are able to think with numbers, and to use numerical ideas to analyze situations and solve a variety of problems. They work with numbers with facility and ease, and demonstrate proficiency with computation.

When children are given instruction at an appropriate level that helps them focus on meaning and relationships, they are

able to accomplish remarkable things. Learning to compute becomes a significant part of their study of mathematics, and helps them develop mathematical thinking and reasoning. In addition to learning how to add, subtract, multiply, and divide, children also learn number relationships, number composition and decomposition, and the principles underlying the structure of the number system. In other words, they are not only competent when dealing with computational problems, but they will also know more mathematics.

Each chapter in the book covers basic concepts and ideas that must be understood if children are going to learn the underlying structure of numbers and how to add, subtract, multiply, and divide with meaning and competence. We look closely at the kinds of thinking children do as they work with these core concepts, and learn to recognize the many stages of thinking that reveal the growth of understanding that occurs over time.

All children progress through certain levels of understanding, albeit at different times, and often without adults' awareness. For example, when children first learn to add numbers like 3 and 4, they think of this as "1 and 1 and 1," and another group of "1 and 1 and 1 and 1." The only way they know to determine the answer is to count all the objects. Later, they will recognize three objects without needing to count them, and will be able to start with 3 and count on to 7. Subsequently, they will more often see that 4 is composed of 3 and 1, and will be able to use the idea that 3 and 3 are 6, so 3 and 4 must be 7. Eventually, they will know that 3 and 4 are 7 without having to think about it. If teachers ignore these stages and just ask the children to memorize the words "three plus four equals seven," they are, in effect, asking them to learn a "song," rather than learn the important relationships these words describe.

At every stage of development, the size of the numbers and the size of the differences between numbers influences what the child is able to understand. That is, a child may be able to see that 5 is contained in 7, but not yet know how 15 is related to 17. Or a child may know the parts of numbers to 4, but not how to figure out the parts of 6.

The level of abstractness also affects what children are able to think about. Children develop a true sense of number by working with real things, rather than with symbols. As children develop their understanding of number, they begin with models, moving them to aid their thinking. They generally advance from being able to think about numbers when they can actually move objects, to thinking about relationships when the model is present but not touched, to thinking about relationships without a physical model. After children can work with an idea at all of these levels, their work with the symbolic representations of this idea will have meaning. Even then, young children rarely reach the level of totally abstract thinking. When working with symbols, they usually need to be thinking of some concrete referent.

Each stage of learning is much more complex than is generally recognized. If we look at counting, for example, we will see that children who are competent counters have integrated several major ideas about counting. These include one-to-one correspondence, inclusion, keeping track, remembering how many, and knowing one more and one less. A child who understands counting will be able to keep track of an unorganized group of objects and will also be able to count out a particular quantity from a larger group of objects. Children who understand place value are able to think of ten as a collection of 10 ones, at the same time they think of the ten

as a unit of 1. They are able to think of 180 ones as 18 tens, or 1 hundred and 8 tens.

This deep understanding of number concepts and relationships does not develop quickly. Children need ongoing and multiple opportunities to develop number sense: to count and compare quantities, to add and subtract, to work with place value in ways that ask them to think and reason, to see relationships, and to make connections. We need to provide children with the kinds of experiences that will help them confront the complexity of these ideas, and encourage them to think and make sense of them.

If we are going to raise achievement in mathematics in ways that allow children to build on what they know, and thus maintain high levels of achievement throughout their schooling, teachers must focus on the mathematics they want children to learn—not on whether they are able to get right answers. Once teachers have identified what children really know and what they need to learn, they will be able to provide appropriate instruction that will give children a solid foundation on which to build, ensuring success for all students.

In the following chapters, we will look at each core topic basic to children's understanding of number. We will identify and describe the Critical Learning Phases within that topic, which must be in place if children are going to build foundational understandings.