The Pythagorean Theorem Tile Set

Guide & Activities

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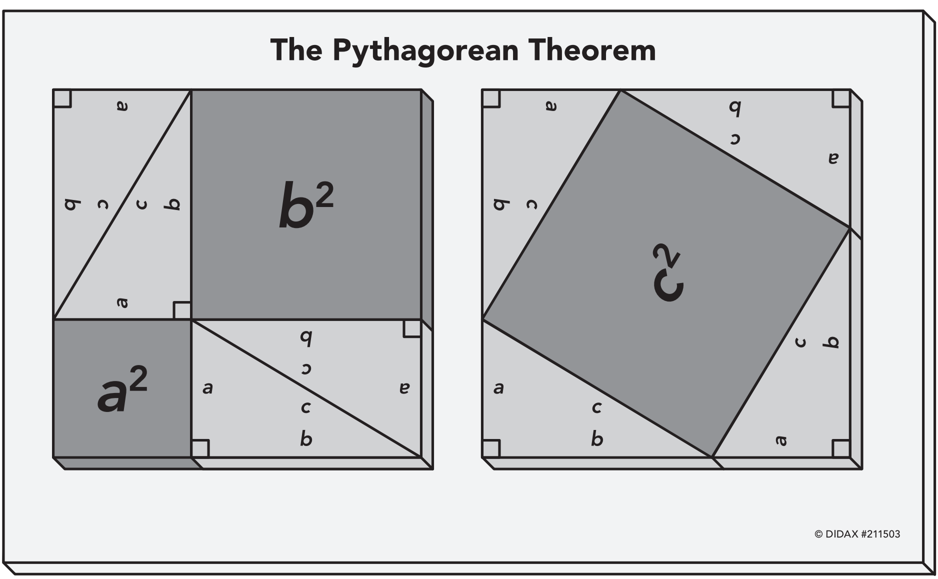

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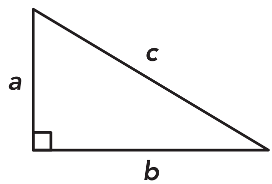
1. Introduction

The Pythagorean Theorem states that in a right triangle the square of the length of the *hypotenuse* is equal to the sum of the squares of the lengths of the *legs*. The *hypotenuse* is the side opposite the right angle and the *legs* are the sides adjacent to the right angle.



$$a^2 + b^2 = c^2$$

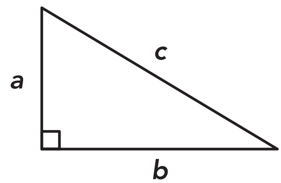
This theorem is named after the Greek philosopher and mathematician Pythagoras (ca. 570 BC – ca. 495 BC). Pythagoras is traditionally credited with the discovery and proof of this theorem, although it is often asserted that knowledge of this theorem existed before Pythagoras.

There are a multitude of proofs of the Pythagorean Theorem, and there is a long and rich mathematical history behind the various proofs of this theorem. Generally, these proofs require a reasonably sophisticated understanding of algebra and geometry and are thus not very accessible to younger students. There exist, however, some proofs of the Pythagorean Theorem that involve a simple rearrangement of shapes. For these proofs by rearrangement, the geometric reasoning is intuitive and the algebraic reasoning is minimal.

These Pythagorean Theorem Tiles provide a hand-held puzzle that gives students an opportunity to discover and visualize the reasoning behind one of the classic and most accessible proofs by rearrangement for the Pythagorean Theorem.

We encourage teachers to use this manipulative with a sense of exploration and discovery. Section 2 outlines an activity with the manipulative that strives to strike the appropriate balance between rigor and discovery, appropriate for grades 6 and above. Section 3 provides an example solution for the activity. Sections 4 and 5 illustrate and apply the Pythagorean Theorem. Section 6 contrasts the Pythagorean Theorem with the results for obtuse and acute triangles.

2. Using Pythagorean Theorem Tiles



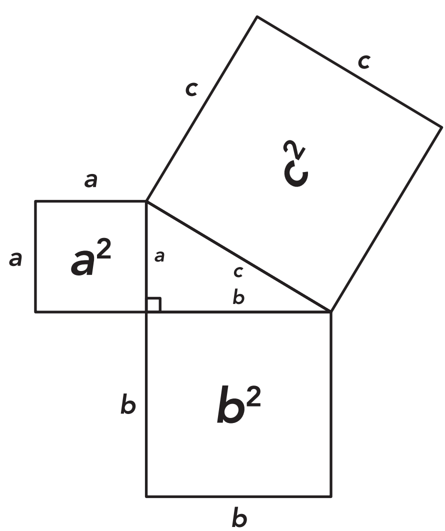
Starting Facts

1. A right triangle is a triangle with a right angle (90 degrees).
2. In a right triangle, the side opposite the right angle is called the *hypotenuse*, with length c. The sides adjacent to the right angle are called the *legs*, with lengths a and b.
3. The area of a square with side length s is s^2.
4. Two objects are *congruent* if they are equal in size and shape.

Materials for the Activity

1. Eight congruent right triangle tiles, each with side lengths a, b, and c.
2. Three square tiles: one with side length a, one with side length b, one with side length c. The square areas are a^2, b^2, c^2, respectively.
3. Two empty congruent square frames, with side lengths $a + b$.

Note that the square tiles of side length a, b, *and* c *fit along the sides of the right triangle tile, as shown.*

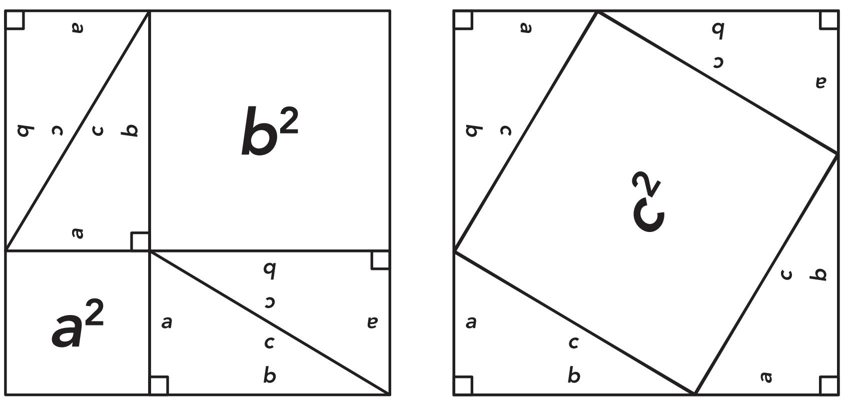


Activity

1. Place 4 triangle tiles, the square tile with area a^2, and the square tile with area b^2 in one of the empty square frames.
2. Place the remaining 4 triangle tiles and the square tile with area c^2 in the other empty square frame.
3. Sketch your results.
4. What do you conclude about the relationship between a^2, b^2, and c^2? Explain.

3. Solutions to Activity Questions 3 and 4

3. Sketch your results.



4. What do you conclude about the relationship between a^2, b^2, and c^2? Explain.

   Let A_{triangle} be the area of any one of the congruent right triangles.

   Since the square frames are congruent, their areas are equal. This means

$$4\,A_{\text{triangle}} + a^2 + b^2 = 4\,A_{\text{triangle}} + c^2.$$

   Subtracting $4\,A_{\text{triangle}}$ from each side of the equation gives

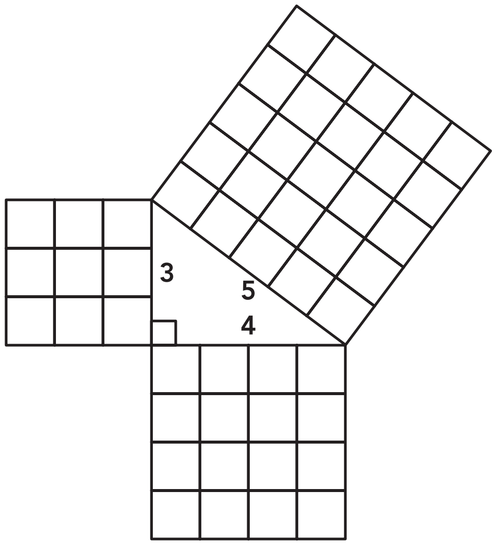
$$a^2 + b^2 = c^2,$$

   which is a statement of the Pythagorean Theorem.

   Note how this solution does not require knowledge of the triangle areas.

4. Illustrating the Pythagorean Theorem

Specific right triangles are often used to illustrate the validity of the Pythagorean Theorem. A classic example is a 3, 4, 5 triangle.



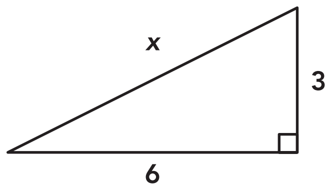
$$a^2 + b^2 = c^2$$
$$3^2 + 4^2 = 5^2$$
$$9 + 16 = 25$$

Other common examples include a 5, 12, 13 triangle (25 + 144 = 169), an 8, 15, 17 triangle (64 + 225 = 289), and appropriate multiples of any of the above triangle side lengths (e.g., a 6, 8, 10 triangle instead of a 3, 4, 5 triangle). These triangles are convenient illustrations of the Pythagorean Theorem because all three side lengths are whole numbers. Having three whole numbers satisfy $a^2 + b^2 = c^2$ is a special occurrence called a *Pythagorean triple*. Usually, when two side lengths of a right triangle are whole numbers, the third side length is not a whole number, and this more typical situation is illustrated in the next section.

5. Applying the Pythagorean Theorem

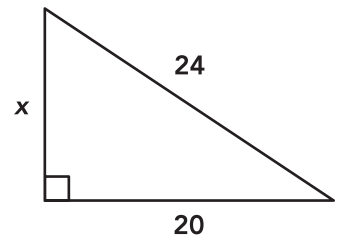
The Pythagorean Theorem allows one to solve for a missing length in a right triangle. In example 1 the hypotenuse is unknown, and in example 2 a leg is unknown.

1.



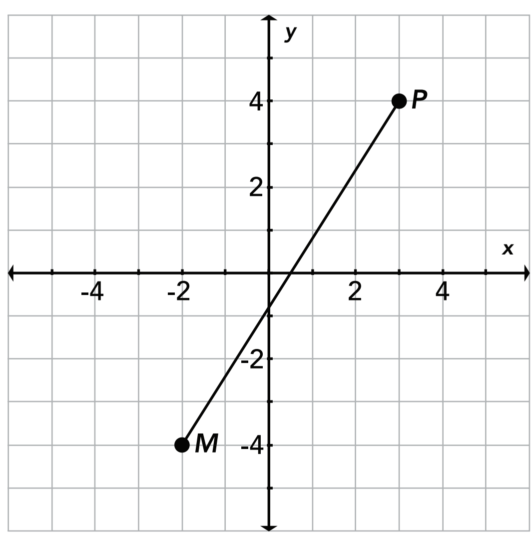
$$a^2 + b^2 = c^2$$
$$3^2 + 6^2 = x^2$$
$$9 + 36 = x^2$$
$$45 = x^2$$
$$\sqrt{45} = x$$
$$x \approx 6.7$$

2.

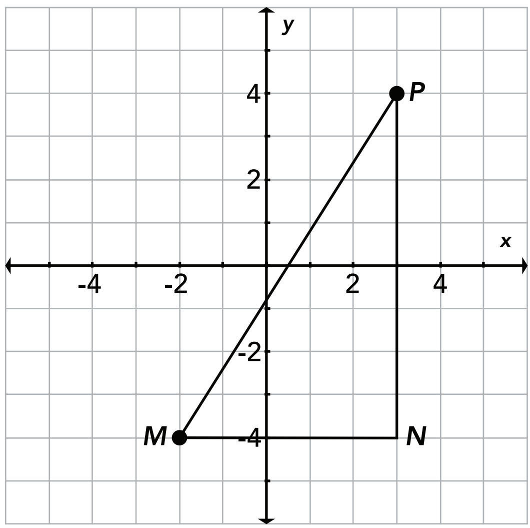


$$a^2 + b^2 = c^2$$
$$x^2 + 20^2 = 24^2$$
$$x^2 + 400 = 576$$
$$x^2 = 176$$
$$x = \sqrt{176}$$
$$x \approx 13.3$$

The ability to solve for a missing length in a right triangle is very useful, since it allows for measurement along an arbitrary direction in the Cartesian plane. For example, the length of line segment $\overline{MP}$ is not obvious since the line segment is not parallel to any grid lines.



However, the Pythagorean Theorem comes to the rescue by envisioning line segment $\overline{MP}$ as the hypotenuse of the right triangle ΔMNP. Since the legs of the right triangle, $\overline{MN}$ and $\overline{NP}$, are parallel to the grid lines, their lengths are easily determined by simply counting unit lengths along the grid lines. Applying the Pythagorean Theorem then allows one to determine the length L of the hypotenuse $\overline{MP}$.



$$a^2 + b^2 = c^2$$
$$8^2 + 5^2 = L^2$$
$$64 + 25 = L^2$$
$$89 = L^2$$
$$\sqrt{89} = L$$
$$L \approx 9.4$$

Much of scientific analysis involves quantities that have direction: force, velocity, displacement, and so on. So the ability to determine magnitude along an arbitrary direction is a cornerstone of mathematical analysis in the physical sciences.

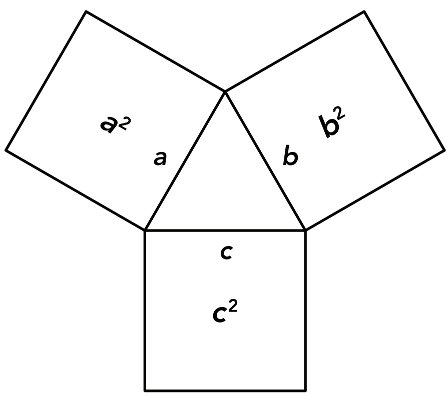
It is interesting to note that quantities like $\sqrt{45}$, $\sqrt{176}$, and $\sqrt{89}$ need to be rounded, since the numbers are irrational and thus their decimal representations continue forever without repetition. Historically, the appearance of irrational numbers in right triangles and other geometric contexts led to a broader understanding of numbers, their classification, and their properties.

6. Comparison with Obtuse and Acute Triangles

The Pythagorean Theorem is so pervasive that students sometime either forget that the theorem only applies to right triangles or are unaware of the relationship between side lengths when a triangle is not a right triangle. We summarize the results below for acute, right, and obtuse triangles. For all triangles, c represents the longest of the three sides, while a and b represent the lengths of the remaining two sides (in either order). The areas of the squares attached to each side help one visualize the relationships stated below.

If $a^2 + b^2 = c^2$, then the triangle is right
(this is the converse of the Pythagorean Theorem).

If $a^2 + b^2 > c^2$, then the triangle is acute (all angles are less than 90º).



If $a^2 + b^2 < c^2$, then the triangle is obtuse (one angle is greater than 90º).